Math 215 — First Midterm

October 10, 2019

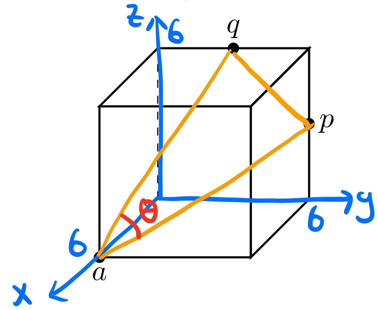
First 3 Letters of Last Name: ☐☐☐ UM Id#: ____________

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam, other than the formula sheet at the end of the exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use no aids (e.g., calculators or notecards) on this exam.
7. **Turn off all cell phones**, remove all headphones, and **place any watch you are using on the desk in front of you**.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	10	
6	15	
7	10	
8	10	
Total	100	

1. [15 points] The sides of the cube below have length six. The points p and q are at the midpoints of their respective edges. Let T be the triangle with vertices a, p, and q.



a. [8 points] What is the area of T?

$a = (6,0,0)$, $p = (0,6,3)$, $q = (0,3,6)$

$$\text{Area}(T) = \frac{1}{2}|\overrightarrow{ap} \times \overrightarrow{aq}|.$$

$\overrightarrow{ap} = (-6,6,3)$, $\overrightarrow{aq} = (-6,3,6)$

$$\overrightarrow{ap} \times \overrightarrow{aq} = (27,18,18)$$

$$\Rightarrow \text{Area}(T) = \frac{1}{2}\sqrt{27^2+18^2+18^2} = \boxed{\frac{9}{2}\sqrt{17}}$$

b. [7 points] If θ is the smallest angle of T, what is $\cos(\theta)$?

$$\cos\theta = \frac{\overrightarrow{ap} \cdot \overrightarrow{aq}}{|\overrightarrow{ap}||\overrightarrow{aq}|} = \frac{72}{9 \cdot 9} = \boxed{\frac{8}{9}}$$

$$\left(\begin{aligned} &\overrightarrow{ap} \cdot \overrightarrow{aq} = (-6)(-6) + 6 \cdot 3 + 3 \cdot 6 = 72 \\ &|\overrightarrow{ap}| = \sqrt{(-6)^2+6^2+3^2} = 9. \\ &|\overrightarrow{aq}| = \sqrt{(-6)^2+3^2+6^2} = 9 \end{aligned}\right)$$

2. [15 points] Indicate if each of the following is true or false by circling the correct answer.

a. [3 points] For any two vectors $\mathbf{u}$ and $\mathbf{v}$, we have $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}| + |\mathbf{v}|$.

True $\boxed{\text{False}}$

$\vec{u} = (1,0,0),\ \vec{v} = (-1,0,0)$

$\Rightarrow |\vec{u}+\vec{v}| = |(0,0,0)| = 0, \quad |\vec{u}|+|\vec{v}| = 1+1 = 2.$

b. [3 points] For any two vectors $\mathbf{u}$ and $\mathbf{v}$, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{v}\times\mathbf{u}|$.

$\boxed{\text{True}}$ False

$|\vec{u}\times\vec{v}| = |-\vec{v}\times\vec{u}| = |\vec{v}\times\vec{u}|$

c. [3 points] If $\mathbf{q}(t)$ is a differentiable vector valued function, then $\frac{d}{dt}|\mathbf{q}(t)| = |\mathbf{q}'(t)|$.

True $\boxed{\text{False}}$

$q(t) = (\cos t, \sin t, 0) \Rightarrow |q(t)| = 1 \Rightarrow \frac{d}{dt}|q(t)| = 0$

$q'(t) = (-\sin t, \cos t, 0) \Rightarrow |q'(t)| = 1$

d. [3 points] Different parameterizations of a space curve C result in identical tangent vectors at a given point on the curve.

True $\boxed{\text{False}}$

$\vec{r_1}(t) = (\cos t, \sin t, 0)$ and $\vec{r_2}(t) = (\cos 2t, \sin 2t, 0)$

parametrize the same circle, but $\vec{r_1}'(t) \neq \vec{r_2}'(t)$.

e. [3 points] If $\mathbf{u}$ and $\mathbf{v}$ are two unit vectors in $\mathbb{R}^3$, then the curve parameterized by $\mathbf{r}(t) = \cos(t)\mathbf{u} + \sin(t)\mathbf{v}$ is a circle.

This is true if $\vec{u}, \vec{v}$ are orthogonal

True $\boxed{\text{False}}$

$\vec{u} = (1,0,0),\ \vec{v} = (1,0,0)$

$\Rightarrow \vec{r}(t) = (\cos t + \sin t, 0, 0)$ is not a circle

as it only moves along the x-axis.

3. [15 points] Beginning at the center of a disk of diameter 50cm, a ladybug crawls along a radius at a speed of 4 cm/sec. Meanwhile, the disk is turning counterclockwise about its center at 1/3 revolutions per second. If we assume that the origin is at the center of the disk, the ladybug initially walks in the direction of the positive x-axis, and the disk is in the plane $z = 0$, then, until the ladybug reaches the edge of the disk, the position of the ladybug is given by

$$\boldsymbol{\ell}(t) = \langle 4t\cos(\omega t), 4t\sin(\omega t), 0\rangle$$

where $\omega = 2\pi/3$.

a. [2 points] At what time would the ladybug reach the edge of the disk?

At the edge : $|\vec{\ell}(t)| = 25$ (radius)

$$|\vec{\ell}(t)| = \sqrt{16t^2\cos^2(wt) + 16t^2\sin^2(wt) + 0} = 4t$$

$$\Rightarrow 4t = 25 \rightsquigarrow \boxed{t = 25/4}$$

b. [4 points] Express, as an integral, how far the ladybug will have travelled during the first six seconds of its journey. Do **not** evaluate the integral.

$$\text{Arc length} = \int_0^6 |\vec{\ell}\,'(t)|\, dt$$

$$\vec{\ell}\,'(t) = (4\cos(wt) - 4wt\sin(wt),\ 4\sin(wt) + 4wt\cos(wt),\ 0)$$

product rule

$$|\vec{\ell}(t)|^2 = (4\cos(wt) - 4wt\sin(wt))^2 + (4\sin(wt) + 4wt\cos(wt))^2$$

$$= 16\cos^2(wt) + 16w^2t^2\sin^2(wt) - 32wt\cos(wt)\sin(wt)$$

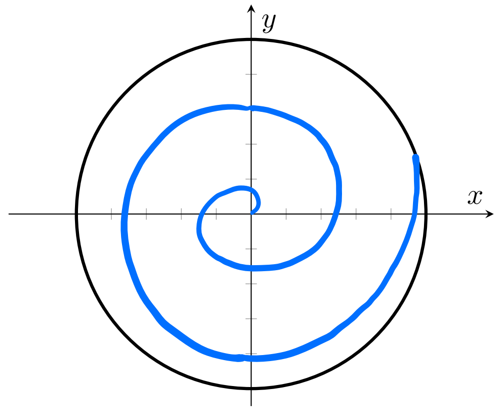
$$+ 16\sin^2(wt) + 16w^2t^2\cos^2(wt) + 32wt\cos(wt)\sin(wt)$$

$$= 16 + 16w^2t^2$$

$$|\vec{\ell}\,'(t)| = \sqrt{16 + 16w^2t^2}$$

$$\Rightarrow \text{Arc length} = \boxed{\int_0^6 \sqrt{16 + 16w^2t^2}\, dt}$$

c. [4 points] Provide a sketch of the path the ladybug would travel.



$\begin{cases} x = 4t\cos(wt), \\ y = 4t\sin(wt) \end{cases}$

$\Rightarrow$ rotation around the origin with increasing radius.

d. [5 points] At time $t = 6$ the ladybug loses its footing and begins to slide across the disk in the direction of the tangent vector $\boldsymbol{\ell}'(6)$. At the same instant, the lady bug begins to fly with acceleration described by $\mathbf{a}(T) = \langle 0, 0, T/10\rangle$ and initial velocity $\boldsymbol{\ell}'(6)$. Here T is the time that has passed since the ladybug lost its footing. Describe the position of the ladybug for $T \geq 0$.

$\vec{r}(T)$: position at time T.

$\vec{r}(0) = \vec{\ell}(6) = (24\cos(6w), 24\sin(6w), 0)$

$\underset{w = \frac{2\pi}{3}}{=} (24\cos(4\pi), 24\sin(4\pi), 0) = (24, 0, 0)$

$\vec{r}\,'(0) = \vec{\ell}\,'(6)$

$\underset{\text{part (b)}}{=} (4\cos(6w) - 24w\sin(6w), 4\sin(6w) + 24w\cos(6w), 0)$

$\underset{w = \frac{2\pi}{3}}{=} (4\cos(4\pi) - 16\pi\sin(4\pi), 4\sin(4\pi) + 16\pi\cos(4\pi), 0)$

$= (4, 16\pi, 0)$

$$\vec{r}''(T) = \vec{a}(T) = (0, 0, \tfrac{T}{10})$$

$$\vec{r}'(T) = \vec{r}'(0) + \int_0^T \vec{r}''(u)\,du$$

$$= (4, 16\pi, 0) + \int_0^T (0, 0, \tfrac{u}{10})\,du$$

$$= (4, 16\pi, 0) + (0, 0, \tfrac{T^2}{20})$$

$$= (4, 16\pi, \tfrac{T^2}{20})$$

$$\vec{r}(T) = \vec{r}(0) + \int_0^T \vec{r}'(u)\,du$$

$$= (24, 0, 0) + \int_0^T (4, 16\pi, \tfrac{u^2}{20})\,du$$

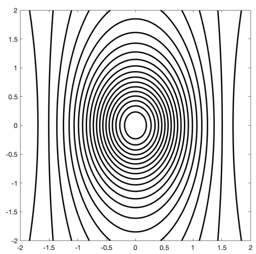
$$= (24, 0, 0) + (4T, 16\pi T, \tfrac{T^3}{60})$$

$$= \boxed{(24 + 4T, 16\pi T, \tfrac{T^3}{60})}$$

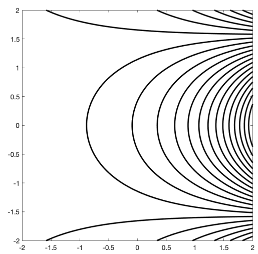
4. [10 points] Match each of the nine sets of level curves below with the appropriate function.

- $\sin(xy)$ ____ ix ____
- $\sin(x)\sin(y)$ ____ viii ____
- $\sin(x) + \sin(y)$ ____ vii ____
- $\cos(x)/(x^2 + y^2 + 1)$ ____ i ____
- $\sin(x)/(x^2 + y^2 + 1)$ ____ vi ____
- $(x^2 + y^2)^{1/2}$ ____ v ____
- $e^x \cos(y)$ ____ ii ____
- $(1 - x^2)(1 - y^2)$ ____ iii ____
- $\sin(x + y)$ ____ iv ____

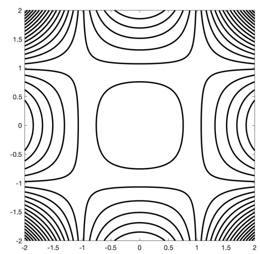
*Explanations on the next page



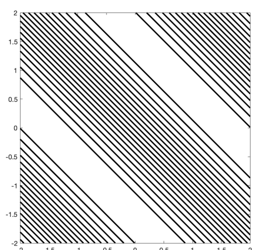
(i)



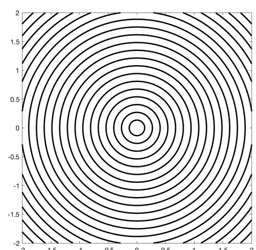
(ii)



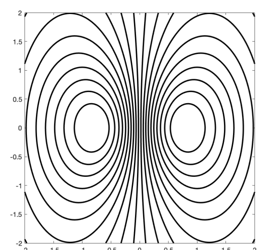
(iii)



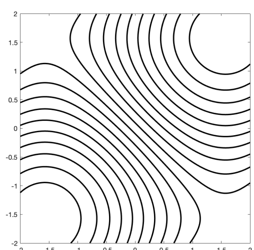
(iv)



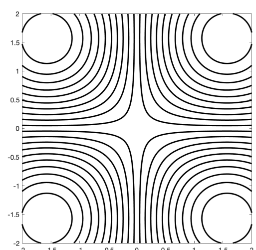
(v)



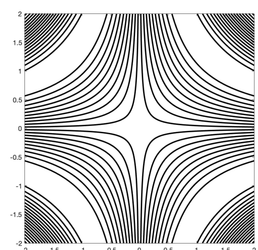
(vi)



(vii)



(viii)

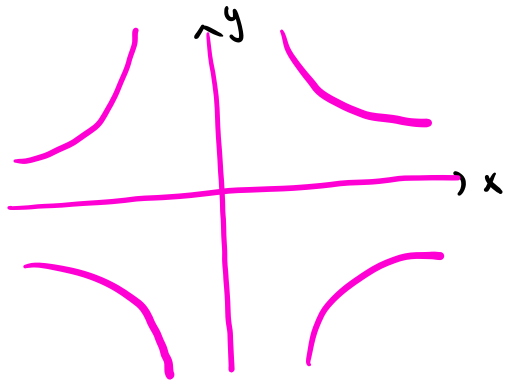


(ix)

Note Level curves at different levels cannot intersect.

Idea: Look at the level 0 and find a contour map which it fits in.

- $\sin(xy) = 0 \Rightarrow xy = 0, \pm\pi, \pm 2\pi, \cdots$

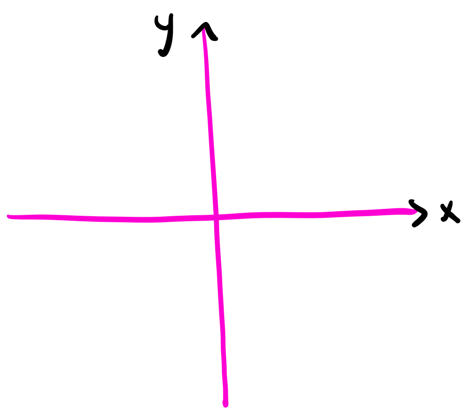


$\Rightarrow$ Match : ix

- $\sin(x)\sin(y) = 0 \Rightarrow \sin(x) = 0$ or $\sin(y) = 0$

$\Rightarrow x = 0, \underline{\pm\pi}, \cdots$ or $y = 0, \underline{\pm\pi}, \cdots$

not in the range of given contour maps

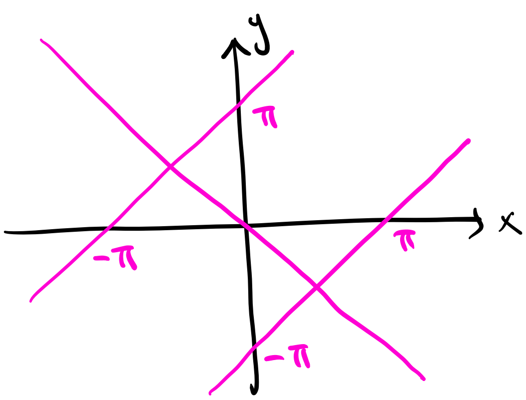


$\Rightarrow$ Match : viii.

- $\sin(x) + \sin(y) = 0 \Rightarrow \sin(x) = -\sin(y)$

$\Rightarrow x = -y,\ y \pm \pi,\ \underline{-y \pm 2\pi}, \cdots$

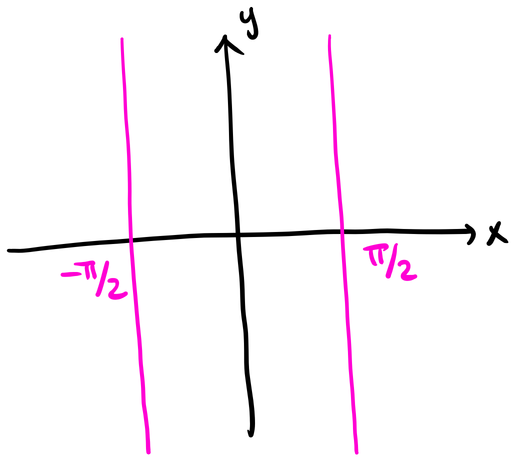
not in the range of given contour maps



$\Rightarrow$ Match : vii.

- $\frac{\cos(x)}{x^2+y^2+1} = 0 \Rightarrow \cos(x) = 0 \Rightarrow x = \pm\frac{\pi}{2}, \underline{\pm\frac{3\pi}{2}}, \cdots$

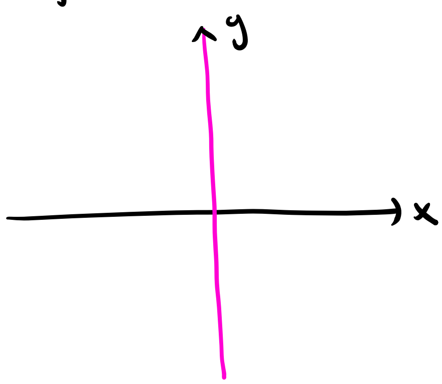
not in the range of given contour maps



$\Rightarrow$ Match : I.

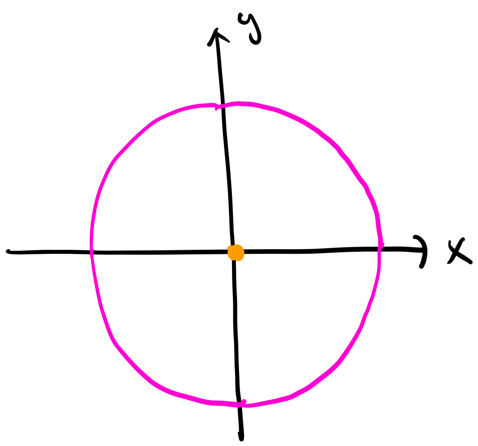
- $\frac{\sin(x)}{x^2+y^2+1} = 0 \Rightarrow \sin(x) = 0 \Rightarrow x = 0, \underline{\pm\pi}, \cdots$

not in the range of given contour maps



- $\sqrt{x^2+y^2} = 0 \Rightarrow x^2+y^2 = 0 \Rightarrow x = y = 0$: the origin

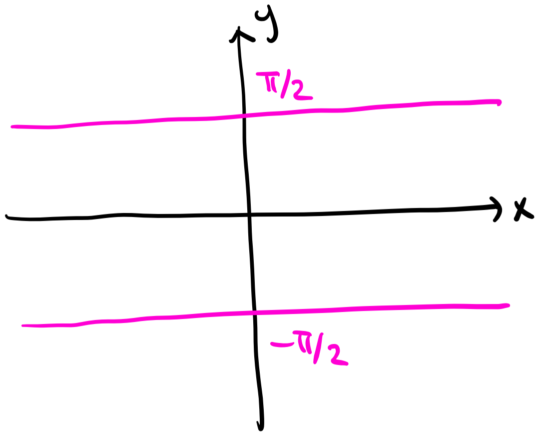
$\sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1$: a circle



$\Rightarrow$ Match : V.

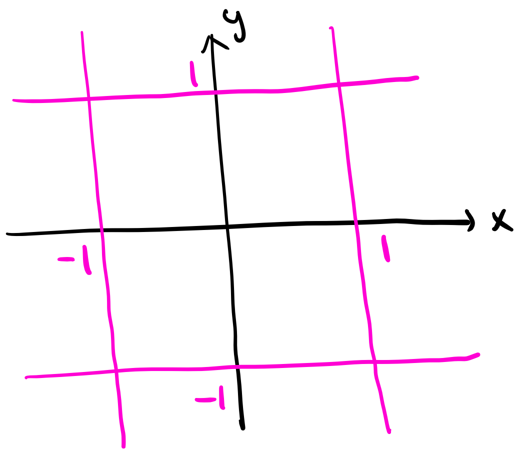
- $e^x \cos(y) = 0 \Rightarrow \cos(y) = 0 \Rightarrow y = \pm\frac{\pi}{2}, \underline{\pm\frac{3\pi}{2}}$

not in the range of given contour maps



$\Rightarrow$ Match : ii

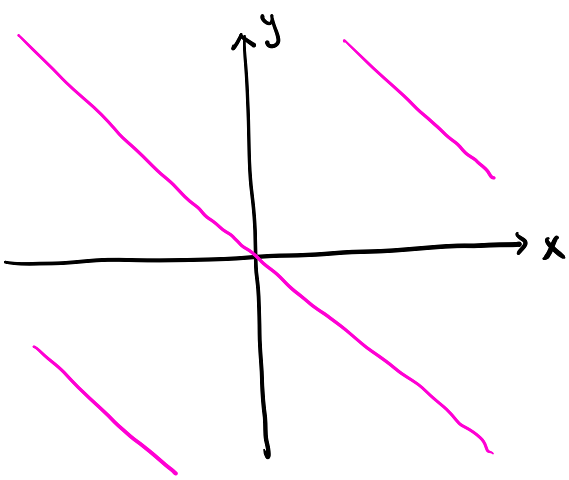
- $(1-x^2)(1-y^2) = 0 \Rightarrow x^2 = 1$ or $y^2 = 1 \Rightarrow x = \pm 1$ or $y = \pm 1$.



$\Rightarrow$ Match : iii

- $\sin(x+y) = 0 \Rightarrow x+y = 0, \pm\pi, \underline{\pm 2\pi}, \cdots$

not in the range of given contour maps



$\Rightarrow$ Match : iv.

5. [10 points] Suppose $g(x, y) = e^{y-x}$.

a. [4 points] Find an equation for the tangent plane to the surface given by the equation $z = g(x, y)$ at the point $(3, 3, 1)$.

$$g_x = \frac{\partial}{\partial x}(e^{y-x}) = -e^{y-x} \Rightarrow g_x(3,3) = -e^{3-3} = -1.$$

$$g_y = \frac{\partial}{\partial y}(e^{y-x}) = e^{y-x} \Rightarrow g_y(3,3) = e^{3-3} = 1.$$

The tangent plane equation is

$$z = g(3,3) + g_x(3,3)(x-3) + g_y(3,3)(y-3)$$

$$\rightsquigarrow \boxed{z = 1 - 1\cdot(x-3) + 1\cdot(y-3)}$$

b. [4 points] Find the linearization $L_g(x, y)$ of the function $g(x, y)$ at the point $(3, 3)$.

$$L_g(x,y) = 1 - 1\cdot(x-3) + 1\cdot(y-3)$$

$$= 1 - x + 3 + y - 3$$

$$= \boxed{1 - x + y}$$

c. [2 points] Use the linear approximation $L_g(x, y)$ to estimate $g(2.9, 3.1)$.

$$g(2.9, 3.1) \approx L_g(2.9, 3.1)$$

$$= 1 - 2.9 + 3.1 = \boxed{1.2}$$

6. [15 points] Find or estimate, depending on the type of data provided, the partial derivative in the x direction **and** the partial derivative in the y direction at the point $(3, 2)$ for each of the following functions.

a. [5 points] For a function f given by the formula $f(x, y) = x^2 e^{xy^2}$

$$f_x = \frac{\partial}{\partial x}(x^2 e^{xy^2}) = 2x e^{xy^2} + x^2 e^{xy^2} \cdot y^2$$

$$f_y = \frac{\partial}{\partial y}(x^2 e^{xy^2}) = x^2 e^{xy^2} \cdot 2xy$$

$$f_x(3,2) = 2 \cdot 3 \cdot e^{3 \cdot 2^2} + 3^2 \cdot e^{3 \cdot 2^2} \cdot 2^2 = \boxed{42e^{12}}$$

$$f_y(3,2) = 3^2 \cdot e^{3 \cdot 2^2} \cdot 2 \cdot 3 \cdot 2 = \boxed{108\, e^{12}}$$

b. [5 points] For a function g described by the data in the table below.

x \ y	1	2	3	4	5
1	7	8	8	9	9
2	13	16	17	18	19
3	21	25	28	31	33
4	29	36	40	45	48
5	37	47	54	62	67

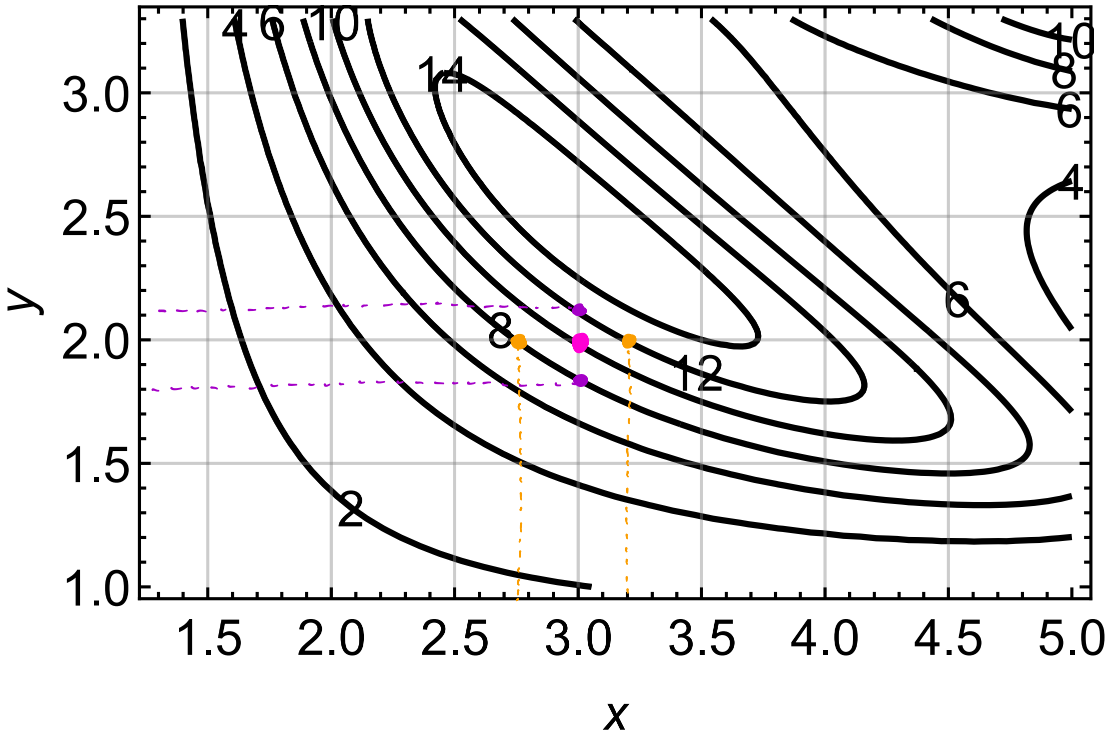
$g_x(3,2) \approx$ Avg. slope from $(2,2)$ to $(4,2)$

$$= \frac{g(4,2) - g(2,2)}{4-2} = \frac{36-16}{2} = \boxed{10}$$

$g_y(3,2) \approx$ Avg. slope from $(3,1)$ to $(3,3)$

$$= \frac{g(3,3) - g(3,1)}{3-1} = \frac{28-21}{2} = \boxed{\frac{7}{2}}$$

c. [5 points] For a function h with level curves as given below.



$h_x(3,2) \approx$ Avg. slope from $(2.75, 2)$ to $(3.2, 2)$

$$= \frac{h(3.2,2) - h(2.75,2)}{3.2 - 2.75}$$

$$= \frac{12-8}{0.45} = \boxed{\frac{80}{9}}$$

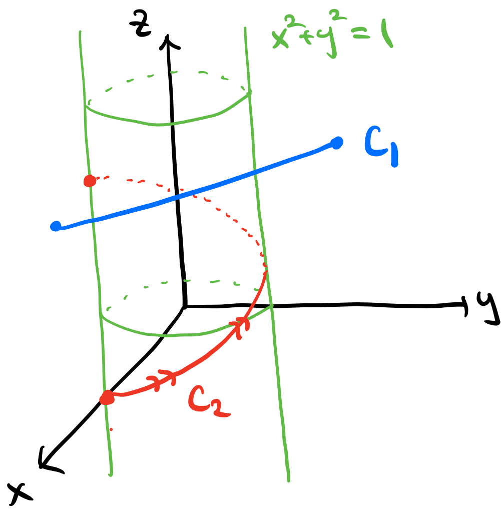
$h_y(3,2) \approx$ Avg. slope from $(3, 1.8)$ to $(3, 2.1)$

$$= \frac{h(3,2.1) - h(3,1.8)}{2.1 - 1.8}$$

$$= \frac{12-8}{0.3} = \boxed{\frac{40}{3}}$$

7. [10 points] The trajectories for two particles are given by $\mathbf{r}_1(t) = \langle t^2 - 2, t^2 - 1, t^2 \rangle$ and $\mathbf{r}_2(t) = \langle \cos(\pi t), \sin(\pi t), t \rangle$ for $0 \le t \le 2$. Let C_1 and C_2 denote the corresponding space curves.

a. [4 points] Graph C_1 and C_2.



$\vec{r_1}(t) = (t^2-2, t^2-1, t^2)$

$= (-2,-1,0) + t^2(1,1,1)$

$\rightsquigarrow$ the line segment from

$\vec{r}(0) = (-2,-1,0)$ to $\vec{r}(2) = (2,3,4)$

$\vec{r_2}(t) = (\cos(\pi t), \sin(\pi t), t)$

$\rightsquigarrow x^2+y^2 = \cos^2(\pi t) + \sin^2(\pi t) = 1$.

$z=t$ increases as t increases

$\rightsquigarrow$ a helix on the cylinder $x^2+y^2=1$

b. [4 points] Do C_1 and C_2 intersect? Justify your answer.

We solve $\vec{r_1}(t) = \vec{r_2}(u)$

$\rightsquigarrow (t^2-2, t^2-1, t^2) = (\cos(\pi u), \sin(\pi u), u)$

$t=1,\ u=1 \Rightarrow \begin{cases} \vec{r_1}(1) = (1^2-2, 1^2-1, 1^2) = (-1,0,1) \\ \vec{r_2}(1) = (\cos(\pi), \sin(\pi), 1) = (-1,0,1) \end{cases}$

$\Rightarrow \vec{r_1}(1) = \vec{r_2}(1)$

$\Rightarrow$ $\boxed{C_1 \text{ and } C_2 \text{ intersect}}$

* Unfortunately, the only way to solve this problem in the scope of Math 215 is by making a guess.

c. [2 points] Do the particles ever collide? Justify your answer.

By part (b), we know $\vec{r}_1(1) = \vec{r}_2(1)$.

$\Rightarrow$ $\boxed{\text{The particles collide}}$

8. [10 points] Suppose P is a plane that contains the points $(0, 3, 0)$, $(-3, 0, 0)$, and $(2, 6, 1)$. Find an equation that describes the sphere centered at the origin that is tangent to P.

Set $A = (0,3,0)$, $B = (-3,0,0)$, $C = (2,6,1)$.

$\overrightarrow{AB} = (-3,-3,0)$, $\overrightarrow{AC} = (2,3,1)$

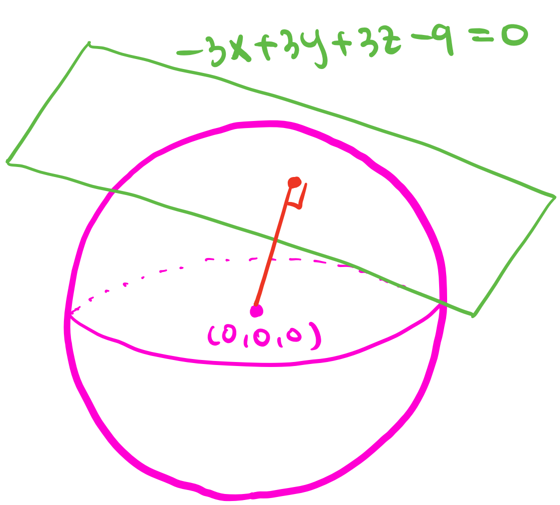
A normal vector of the plane P is

$\overrightarrow{AB} \times \overrightarrow{AC} = (-3,-3,0) \times (2,3,1) = (-3,3,3)$.

The plane equation is

$-3(x-0) + 3(y-3) + 3(z-0) = 0$

$\rightsquigarrow -3x + 3y + 3z - 9 = 0$.



The radius of the sphere is the distance from $(0,0,0)$ to the plane $-3x+3y+3z-9=0$.

$$\frac{|-3\cdot 0 + 3\cdot 0 + 3\cdot 0 - 9|}{\sqrt{(-3)^2+3^2+3^2}} = \sqrt{3}.$$

$\Rightarrow$ The sphere equation is $\boxed{x^2+y^2+z^2=3}$.

You may use this page for scratch work.

This sheet will not be graded. Do not turn it in.

- $\sin^2(x) + \cos^2(x) = 1$, $\cos(2x) = \cos^2(x) - \sin^2(x)$, $\sin(2x) = 2\sin(x)\cos(x)$
- $\sin^2(x) = \dfrac{1-\cos(2x)}{2}$, $\cos^2(x) = \dfrac{1+\cos(2x)}{2}$
- $\cos(\pi/3) = 1/2$, $\sin(\pi/3) = \sqrt{3}/2$, $\cos(\pi/4) = \sqrt{2}/2$, $\sin(\pi/4) = \sqrt{2}/2$, $\cos(\pi/6) = \sqrt{3}/2$, $\sin(\pi/6) = 1/2$, $\cos(0) = 1$, $\sin(0) = 0$.
- $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$.
- Volume of the parallelepiped determined by the vectors $\mathbf{v_1} = \langle a, b, c\rangle$, $\mathbf{v_2} = \langle d, e, f\rangle$, and $\mathbf{v_3} = \langle g, h, i\rangle$ is $|\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})| =$ absolute value of $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$
- Distance from a point (a, b, c) to a plane $Ax + By + Cz + D = 0$ is $\frac{|Aa+Bb+Cc+D|}{\sqrt{A^2+B^2+C^2}}$.
- The circumference of a circle of radius a is $2\pi a$.
- The area of a disk of radius a is πa^2.
- The volume of a right circular cylinder of radius a and height h is $\pi a^2 h$.